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ECE 204 *Numerical methods*

The method of successive over-relaxation

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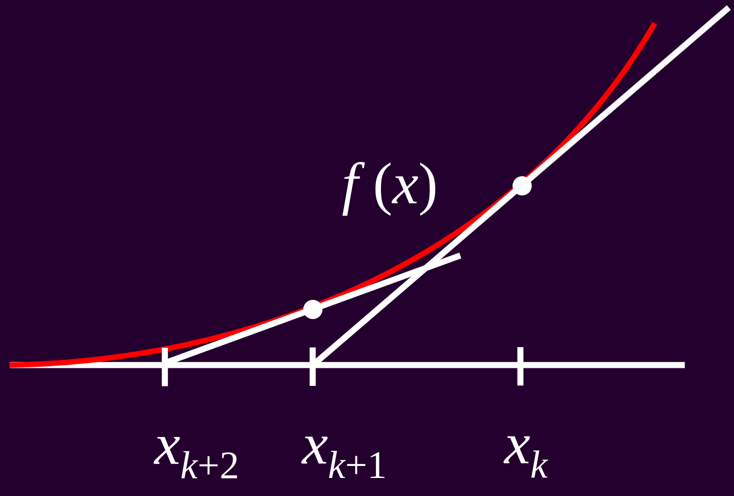
Introduction

- In this topic, we will
  - Describe the concept of over-relaxation
  - Apply this concept to the Jacobi and Gauss-Seidel methods
  - Consider the change to the implementation

Newton's method and double roots

- Suppose we are converging on a double root using Newton's method
  - This causes a problem, as convergence will slow
  - The numerator $f(x)$ goes to zero faster than the denominator $f^{(1)}(x)$



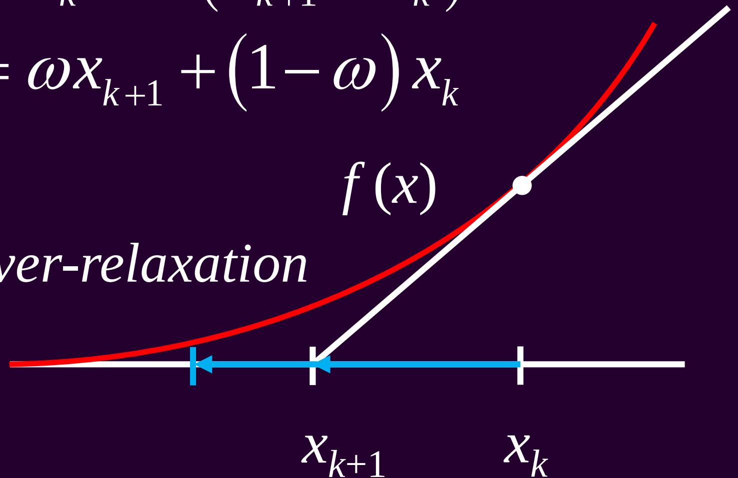
Over-relaxation

- Suppose we have x_k and x_{k+1} where $x_{k+1} \leftarrow x_k + \Delta x_k$
  - Can we change how much we move in the direction?
  - Given x_k, move $x_{k+1} \leftarrow x_k + \omega\,\Delta x_k$
  - Thus, we can calculate x_{k+1} but then set

$$\begin{aligned} x_{k+1} &\leftarrow x_k + \omega \Delta x_k \\ &= x_k + \omega\left(x_{k+1} - x_k\right) \\ &= \omega x_{k+1} + \left(1 - \omega\right) x_k \end{aligned}$$

Note $\omega + (1 - \omega) = 1$

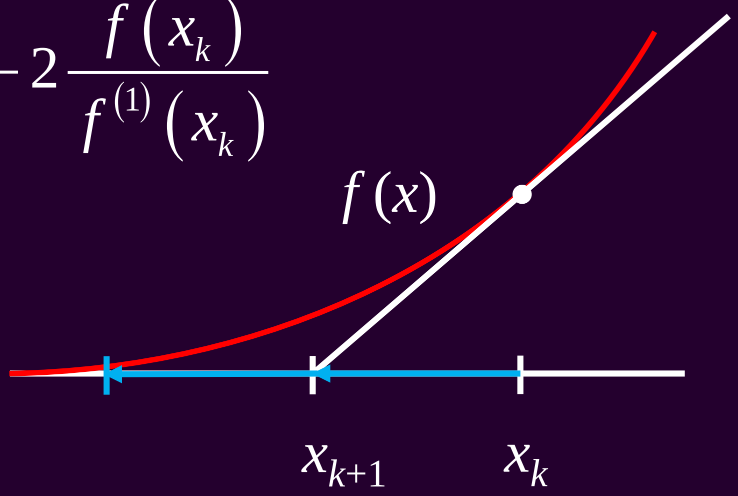
  - This is called *over-relaxation*



Over-relaxation

- Thus, having calculated x_{k+1}
  - If $\omega < 1$, $\omega x_{k+1} + (1 - \omega) x_k$ does not move as far as x_{k+1}
  - If $\omega = 1$, $\omega x_{k+1} + (1 - \omega) x_k = x_{k+1}$ does not change
  - If $\omega > 1$, $\omega x_{k+1} + (1 - \omega) x_k$ moves further than x_{k+1}
- If ω is too large , we will overshoot the target
  - For Newton's method and a double root, $\omega = 2$ is ideal:

$$x_{k+1} \leftarrow x_k - 2\frac{f(x_k)}{f^{(1)}(x_k)}$$



Jacobi and over-relaxation

- We can modify the Jacobi method to include this additional push
  - The ideal size of ω depends on the matrix, but it's reasonable to start with values slightly larger that 1
  - If the previous value is $\mathbf{u}_k$ and our next approximation is

$$\mathbf{u}_{k+1} \leftarrow A_{\text{diag}}^{-1}\left(\mathbf{v} - A_{\text{off}}\mathbf{u}_k\right)$$

  - Then update

$$\mathbf{u}_{k+1} \leftarrow \omega\mathbf{u}_{k+1} + (1-\omega)\mathbf{u}_k$$

Gauss-Seidel and over-relaxation

- We can modify the Gauss-Seidel method to include this additional push
  - The ideal size of ω depends on the matrix, but it's reasonable to start with values slightly larger that 1
  - If the previous value is $\mathbf{u}_{k;\,i}$ and our next approximation is

$$\mathbf{u}_{k+1;\,i} \leftarrow \frac{1}{a_{i,i}}\left(v_i - A_{\text{off; i},\cdots}\mathbf{u}_{k+1}\right)$$

  - Then update $\mathbf{u}_{k+1;\,i} \leftarrow (1+\omega)\mathbf{u}_{k;\,i} - \omega\mathbf{u}_{k+1;\,i}$

Implementation

- Suppose we are solving $A\mathbf{u} = \mathbf{v}$ and $\mathbf{u}$ is the current approximation:

```
for k = 1:max_iterations
    u_old = u;

    for i = 1:n
        u(i) = v(i);

        for j = [1:i-1, i+1:n]
            u(i) = u(i) - A(i,j)*u(j);
        end

        u(i) = (1 - omega)*u_old(i) + omega*u(i)/A(i,i);
    end

    if norm( u - u_old ) < eps_step
        return; // returns 'u'
    end
end
```

This used to be
`u(i) = u(i)/A(i,i);`

Summary

- Following this topic, you now
  - Understand the idea behind over-relaxation
  - Are aware this is a *push* in the direction our next approximation moves
  - Understand the ideal value of ω depends on the matrix
  - Understand this can speed up convergence,
    but it can also slow it down
    - Useful for simulations
  - Are aware that this is a trivial change to the code
    - Only O(1) additional operations per entry

References

[1] https://en.wikipedia.org/wiki/Successive_over-relaxation

Acknowledgments

Chuhan Xiang for detecting a typo (missing the word “not”) on Slide 5.

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`. Mathematical equations are prepared in MathType by Design Science, Inc.

Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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for more information.

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